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An Equation for the Dissipation Rate Correlation and Its Implications for the Intermittency Exponent μ in Turbulence

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We derive an equation satisfied by the dissipation rate correlation function, $\langle \epsilon(\vec{x}+\vec{r}, t+\tau)\, \epsilon(\vec{x}, t) \rangle$ for the homogeneous, isotropic state of fully-developed turbulence from the the Navier–Stokes equation. In the equal time limit we show that the equation leads directly to two intermittency exponents $\mu_1 = 2 - \zeta_6$ and $\mu_2 = z_4'' - \zeta_4$, where the ζ's are exponents of velocity structure functions and z_4'' is a dynamical exponent characterizing the fourth order structure function. We discuss the contributions of the pressure terms to the equation and the consequences of hyperscaling.

KEY WORDS: Turbulence exponents; Navier–Stokes equations.

1. INTRODUCTION

The statistical properties of the energy dissipation rate $\epsilon(\vec{x}, t)$ defined by

$$\epsilon(\vec{x}, t) = \frac{\nu}{2} \sum_{i,j} (\partial_i u_j + \partial_j u_i)^2 \tag{1}$$

where $\partial_i = \partial/\partial x_i$ have played a crucial role in our understanding of fully-developed turbulence in incompressible fluids.[(1–3)] In the original Kolmogorov theory ϵ is replaced by $\langle \epsilon \rangle$ and the spatial fluctuations are ignored; the effect of fluctuations of ϵ pointed out by Landau have been explored in the context of the lognormal model and its multifractal

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generalizations.[4] The intermittent behavior of turbulent fluctuations is reflected in the power-law behavior of the equal-time correlations of ϵ:

$$\langle \epsilon(\vec{x})\, \epsilon(\vec{x}+\vec{r}) \rangle \sim (r/L)^{-\mu} \tag{2}$$

where L is a length scale characteristic of the large-scale flow and $r = |\vec{r}|$ belongs to the inertial range. Simple dimensional analysis, noting that the dimension of ϵ is V^3/L, yields the identification $\mu = 2 - \zeta_6$; the exponents of the qth order (longitudinal) structure function, ζ_q, is defined by

$$S_q \equiv \langle [\delta\vec{u} \cdot \hat{r}]^q \rangle \sim (r/L)^{\zeta_q} \tag{3}$$

where $\delta\vec{u} = \vec{u}(\vec{x}+\vec{r}, t) - \vec{u}(\vec{x}, t)$. Within the original Kolmogorov theory $\zeta_6 = 2$ and consequently, $\mu = 0$. Thus the deviation of μ from zero is a measure of the degree of intermittency and is an important quantity for understanding fully developed turbulence. The breakdown of simple Kolmogorov scaling has been studied experimentally and a review of the experiments[5] gives a "best" estimate for μ of 0.25 ± 0.05 which is consistent with the experimentally measured value[1] of $\zeta_6 \approx 1.8$. In this Letter we provide a simple and direct derivation of the scaling relations satisfied by μ for the Navier–Stokes equation by deriving an equation satisfied by the dissipation rate correlations. The equation for the dissipation rate contains not only a contribution from a second spatial derivative of an appropriate sixth-order structure function, which yields the above-mentioned value for $\mu = 2 - \zeta_6$, but also that from a second temporal derivative of a fourth-order structure function; there are, in addition, pressure-dependent terms which are exhibited explicitly and no other velocity-dependent terms. The use of *dynamical* structure functions, i.e., in Eq. (3) $\delta\vec{u} = \vec{u}(\vec{x}+\vec{r}, t+\tau) - \vec{u}(\vec{x}, t)$, is key to our derivation.[6] Both spatial and temporal derivatives of the dynamical structure functions occur naturally and the equal time limit of the derivatives is related to the correlation of the energy dissipation rate. An earlier paper by us[7] provided a justification for the same result in the context of the stochastic Burgers equation and argued entirely by analogy how the same results would arise in the Navier–Stokes case. The heuristic discussion depended on the mathematical similarity of some of the terms which occur in detailed dynamical equations for the structure functions and on an equation for the dissipation-rate correlations which was not explicitly Galilean-invariant. In this paper we give a manifestly Galilean invariant equation for the dissipation rate equation from which the dominant exponents arise directly. We discuss moreover the contributions of the pressure terms explicitly and argue that they do not yield more dominant behavior. In the next section we present the derivation of the main equation; in the following section we discuss the contributions of the different terms to the behavior of the dissipation rate correlation in the equal time limit, in particular, the terms which depend on the pressure, and the consequences of hyperscaling. Many of the details of the calculations are relegated to Appendices A–E.

2. DERIVATION OF THE EQUATION FOR DISSIPATION-RATE CORRELATION

We consider the Navier–Stokes equation for an incompressible velocity field $\vec{u}(\vec{x}, t)$,

$$\partial_t u_i(\vec{x}, t) + u_j \, \partial u_i / \partial x_j = \nu \, \nabla^2 u_i - \partial \tilde{p} / \partial x_i + f_i \tag{4}$$

where $\tilde{p} = p/\rho$ and ρ is the constant density. We have employed the summation convention of summing over repeated indices. The system is driven by a Gaussian, stochastic driving force $\vec{f}(\vec{x}, t)$ with zero mean and variance given by

$$\langle \hat{f}_i(\vec{k}, t) \, \hat{f}_j(\vec{k}', t) \rangle = P_{ij}(\vec{k}) \, D(k) \, \delta_{\vec{k}+\vec{k}', \vec{0}} \, \delta(t - t') \tag{5}$$

where $P_{ij}(\vec{k})$ is the transverse projection operator given by $\delta_{ij} - (k_i k_j / k^2)$. The noise covariance $D(k)$ is assumed to be peaked around $k_0 \sim 1/L$ with a narrow width. The detailed form of the noise correlation is not important; the stochastic forcing maintains a fully-developed turbulent state and allows one to define averages as noise ensemble averages. We will find it useful to define the quantity

$$\epsilon_{ij} \equiv \nu \, \partial_\ell u_i \, \partial_\ell u_j. \tag{6}$$

The dissipation rate ϵ (cf. Eq. (1)) of an incompressible fluid obeys the relation

$$\epsilon = \epsilon_{ii} - \nu \nabla^2 \tilde{p}. \tag{7}$$

We remark that $\langle \epsilon_{ij} \rangle \propto \delta_{ij}$ in isotropic turbulence.

We first describe an illustrative calculation which exemplifies our method; the complete calculation is discussed later. We employ the notation $\vec{x} = \vec{R} + (1/2)\,\vec{r}$, $t = T + (1/2)\,\tau$ and $\vec{x}' = \vec{R} - (1/2)\,\vec{r}$, $t' = T - (1/2)\,\tau$.

Multiplying the Navier–Stokes equation for $u_i = u_i(\vec{x}, t)$ by u_i, summing over i, and using $\nu u_i \nabla^2 u_i = \nu \nabla^2 (u^2/2) - \epsilon_{ii}$ one finds

$$-\epsilon_{ii} + \nu \nabla^2 (u^2/2) + \vec{f} \cdot \vec{u} = u_i\, \partial_t u_i + u_i u_l\, \partial_l u_i - u_i\, \partial_i \tilde{p}. \tag{8}$$

We write a similar equation for $u_i' = u_i(\vec{x}', t')$ and multiply the two equations and average over the homogenous, steady state of isotropic turbulence. This yields

$$\begin{aligned} \langle \epsilon_{ii} \epsilon'_{jj} \rangle &- \frac{\nu}{2} \nabla_r^2 \langle \epsilon_{ii} u'^2 + \epsilon'_{ii} u^2 \rangle + \frac{\nu^2}{4} \nabla_r^2 \nabla_r^2 \langle u^2 u'^2 \rangle + \text{noise terms} \\ &= -\frac{1}{4} \frac{\partial^2}{\partial \tau^2} \langle u^2 u'^2 \rangle - \frac{1}{2} \frac{\partial^2}{\partial \tau\, \partial r_i} \langle (u_i + u_i')\, u^2 u'^2 \rangle \\ &\quad - \frac{1}{4} \frac{\partial^2}{\partial r_i\, \partial r_j} \langle u_i u_j u^2 u'^2 \rangle + \text{pressure terms}. \end{aligned} \tag{9}$$

The second and third terms on the left-hand side are negligible in the inertial range; this is evident since the correlation function $\langle u^2 u'^2 \rangle$ is finite in the inertial range and when multiplied by ν vanishes as $\nu \to 0$. In the second term one factor of ν has been absorbed into obtaining ϵ_{ii}, a finite quantity, and again the finite expression $\langle \epsilon_{ii} u'^2 + \epsilon'_{ii} u^2 \rangle$ is multiplied by ν and thus vanishes in the $\nu \to 0$ limit.[8] We immediately see that the velocity terms on the right-hand side of the above equation consist of a second temporal derivative of a fourth-order structure function and the second spatial derivative of a sixth-order structure function. Of course, the equation is not manifestly form-invariant under Galilean transformations. We also have to understand the role of the cross spatio-temporal derivative and the pressure terms. We address these issues using a different version of the above equation which is manifestly Galilean-invariant. We have not displayed the noise and pressure terms since the above calculation merely illustrates the simple steps involved in the derivation.

We present next the derivation of the equation for the dissipation rate correlation function. It is useful to introduce the difference variable $\delta u_i \equiv u_i(\vec{x}, t) - \vec{u}_i(\vec{x}', t')$. We proceed from the Navier–Stokes equation in two different ways and combine the results appropriately. First, we consider the product of the Navier–Stokes equation for u_i with δu_i and sum over i which yields

$$\delta u_i (\nu \nabla^2 u_i + f_i) = \delta u_i \left(\frac{D u_i}{Dt} + \partial_i \tilde{p} \right) \tag{10}$$

where we have used the summation convention. In the preceding the total or convective derivative, $\partial u_i/\partial t + u_j\, \partial u_i/\partial x_j$, is denoted by Du_i/Dt. We multiply the above equation by the corresponding equation for the primed variable to obtain

$$\delta u_i(\nu \nabla^2 u_i + f_i)\,\delta u_j(\nu \nabla'^2 u'_j + f'_j) = \delta u_i\left(\frac{Du_i}{Dt} + \partial_i \tilde{p}\right)\delta u_j\left(\frac{Du'_j}{Dt'} + \partial'_j \tilde{p}'\right). \tag{11}$$

We average over the homogeneous, steady state of turbulence. On the left-hand side in addition to the noise terms, $-\langle \epsilon_{ii}\epsilon'_{jj} + \epsilon_{ij}\epsilon'_{ij}\rangle$ are the only terms which survive in the inertial range. The derivation of this result is outlined in Appendix A. Next we consider the Navier–Stokes equation for u_i multiplied by δu_j and the corresponding equation for u'_i (multiplied by δu_j) and take the product of the two equations summed over i and j; this yields

$$\delta u_j(\nu \nabla^2 u_i + f_i)\,\delta u_j(\nu \nabla'^2 u'_i + f'_i) = \delta u_j\left(\frac{Du_i}{Dt} + \partial_i \tilde{p}\right)\delta u_j\left(\frac{Du'_i}{Dt'} + \partial'_i \tilde{p}'\right). \tag{12}$$

We then add twice Eq. (11) to Eq. (12). In the resulting equation we obtain for the noise-independent parts on the left-hand side using the results of Appendix A,

$$-2\langle \epsilon_{ii}\epsilon'_{jj} + 2\epsilon_{ij}\epsilon'_{ij}\rangle - \frac{\nu^2}{4}(\nabla_r^2)^2\langle(\delta\vec{u}\cdot\delta\vec{u})^2\rangle$$
$$+\nu\nabla_r^2\langle(\epsilon_{ii} + \epsilon'_{ii})\,\delta\vec{u}\cdot\delta\vec{u} + 2(\epsilon_{ij} + \epsilon'_{ij})\,\delta u_i\,\delta u_j\rangle. \tag{13}$$

The only terms which survive in the inertial range are $-2\langle\epsilon_{ii}\epsilon'_{jj}\rangle - 4\langle\epsilon_{ij}\epsilon'_{ij}\rangle$. The right-hand side of the resulting equation contains the following terms, which we denote by C_1, apart from the pressure-dependent contributions:

$$C_1 \equiv 2\left\langle \delta u_i \frac{Du_i}{Dt}\,\delta u_j \frac{Du'_j}{Dt'}\right\rangle + \left\langle \delta u_j \frac{Du_i}{Dt}\,\delta u_j \frac{Du'_i}{Dt'}\right\rangle. \tag{14}$$

We can show after some algebraic manipulations (Appendix B provides some details) involving kinematic relations that the above expression for C_1 is equal to the following

$$C_1 = -(1/16)(\partial^2/\partial r_i\,\partial r_j)\langle\delta u_i\,\delta u_j(\delta\vec{u}\cdot\delta\vec{u})^2\rangle + (1/4)(\hat{D}^2/\hat{D}\tau^2)\langle(\delta\vec{u}\cdot\delta\vec{u})^2\rangle \tag{15}$$

where we have defined

$$\frac{\hat{D}^2}{\hat{D}\tau^2}\langle f\rangle \equiv \frac{\partial^2}{\partial\tau^2}\langle f\rangle + \frac{\partial^2}{\partial\tau\,\partial r_i}\langle (u_i+u_i')\,f\rangle + \frac{1}{4}\frac{\partial^2}{\partial r_i\,\partial r_j}\langle (u_i+u_i')(u_j+u_j')\,f\rangle \tag{16}$$

the Galilean-invariant second derivative with respect to τ of the expectation value of a function of velocity differences. This shows explicitly the two key terms, the second spatial derivative of the sixth-order structure function and the (Galilean-invariant) second time derivative of a fourth-order structure function. We have evaluated the pressure terms and find

$$\begin{aligned}
&2\langle \epsilon_{ii}\epsilon_{jj}' + 2\epsilon_{ij}\epsilon_{ij}'\rangle + \text{noise terms}\\
&\quad = \frac{1}{16}\frac{\partial^2}{\partial r_i\,\partial r_j}\langle \delta u_i\,\delta u_j (\delta\vec{u}\cdot\delta\vec{u})^2\rangle - \frac{1}{4}\frac{\hat{D}^2}{\hat{D}\tau^2}\langle (\delta\vec{u}\cdot\delta\vec{u})^2\rangle\\
&\qquad + \langle 2\,\delta u_i\,\delta u_j\,\partial_i\tilde{p}\,\partial_j'\tilde{p}' + \delta u_i\,\delta u_i\,\partial_j\tilde{p}\,\partial_j'\tilde{p}'\rangle - \frac{\hat{D}}{\hat{D}\tau}\langle (\partial_i\tilde{p} + \partial_i'\tilde{p}')\,\delta u_i\,\delta\vec{u}\cdot\delta\vec{u}\rangle\\
&\qquad + \frac{1}{2}\partial_{r_\ell}\langle\,\delta u_\ell\,\delta\vec{u}\cdot\delta\vec{u}\,\delta u_i(\partial_i\tilde{p} - \partial_i'\tilde{p}')\rangle.
\end{aligned} \tag{17}$$

We emphasize that apart from terms negligible in the inertial range in the limit $\nu\to 0$ (most of which are displayed in Appendix A) the above equation is exact. To complete our discussion we must evaluate the noise terms and discuss the contribution of the pressure terms in Eq. (17). It is useful to extract the parts which survive in the inertial range in the first two terms of Eq. (17). It is easy to show from Eq. (7) that

$$\langle \epsilon_{ii}\epsilon_{jj}'\rangle = \langle \epsilon\epsilon'\rangle + \nu\nabla_r^2\langle \epsilon\tilde{p}' + \epsilon'\tilde{p}\rangle + \nu^2\nabla_r^2\nabla_r^2\langle \tilde{p}\tilde{p}'\rangle \tag{18}$$

which is equal to $\langle\epsilon\epsilon'\rangle$ in the inertial range. The other two terms in Eq. (18) are finite correlation functions multiplied by powers of ν and hence, vanish in the inertial range. We consider $\langle \epsilon_{ij}\epsilon_{ij}'\rangle$ next; we observe that the diagonal elements ($i=j$) of $\langle \partial_i u_\ell\,\partial_j u_\ell\rangle$ yield the most singular terms and only these survive in the $\nu\to 0$ limit which cuts off the short-distance singularities. With this observation we see that only the terms with $i=j$ contribute in $\langle \epsilon_{ij}\epsilon_{ij}'\rangle$; using the isotropy of the turbulent state we then find that the first two terms on the left-hand side of Eq. (17) reduce to $(10/3)\langle\epsilon\epsilon'\rangle$ in the inertial range.

The equation for the dissipation rate correlation, Eq. (17) depends on both $\vec{r}$ and τ. The noise terms can be simplified in the $\tau\to 0$ limit using the

Donsker–Novikov–Varadhan result for Gaussian noise[9] and the non-trivial term in the $\tau \to 0^+$ limit is precisely $(10/3)\langle\epsilon\rangle^2$. Some of the details of the derivation of this result can be found in Appendix C. Thus the left-hand side of Eq. (17) reduces in the equal-time limit and in the inertial range to $(10/3)[\langle\epsilon\epsilon'\rangle - \langle\epsilon\rangle^2]$. Finally, we have the complete equation for the equal-time dissipation rate correlation in the inertial range

$$\begin{aligned}\frac{10}{3}[\langle\epsilon\epsilon'\rangle-\langle\epsilon\rangle^2] = & \frac{1}{16}\frac{\partial^2}{\partial r_i\,\partial r_j}\langle\delta u_i\,\delta u_j(\delta\vec{u}\cdot\delta\vec{u})^2\rangle - \frac{1}{4}\frac{\hat{D}^2}{\hat{D}\tau^2}\langle(\delta\vec{u}\cdot\delta\vec{u})^2\rangle \\ & + \langle 2\,\delta u_i\,\delta u_j\,\partial_i\tilde{p}\,\partial'_j\tilde{p}' + \delta u_i\,\delta u_i\,\partial_j\tilde{p}\,\partial'_j\tilde{p}'\rangle \\ & - \frac{\hat{D}}{\hat{D}\tau}\langle(\partial_i\tilde{p}+\partial'_i\tilde{p}')\,\delta u_i\,\delta\vec{u}\cdot\delta\vec{u}\rangle \\ & + \frac{1}{2}\partial_{r_\ell}\langle\,\delta u_\ell\,\delta\vec{u}\cdot\delta\vec{u}\,\delta u_i(\partial_i\tilde{p}-\partial'_i\tilde{p}')\rangle. \qquad (19)\end{aligned}$$

This is the first time, to the best of our knowledge, that this equation for the dissipation rate correlation, with each term on the right-hand side manifestly Galilean-invariant, appears in the literature.

3. DISCUSSION OF RESULTS

The equal-time behavior of $\langle\epsilon\epsilon'\rangle$ is thus determined by the $\tau \to 0$ limit of the terms occurring on the right-hand side of Eq. (19). There are two terms which do not involve the pressure and these yield the two exponents $\mu_1 = 2-\zeta_6$ and $\mu_2 = z''_4 - \zeta_4$. The first arises from the second spatial derivative of the sixth-order structure function. The second arises from the $\tau \to 0$ limit of the second temporal derivative of the fourth-order dynamical structure function by postulating that the expansion of $S_4(r,\tau)$ in powers of τ contains $\tau/r^{z'_4}$ and $\tau^2/r^{z''_4}$ with different dynamical exponents characterizing different powers of τ allowing for multifractality in temporal behavior. The existence of Eq. (17) allows us to provide a transparent derivation directly from the Navier–Stokes equation of the two exponents characterizing dissipation-rate correlations, one which depends purely on the static structure function exponent $(2-\zeta_6)$ and the other which involves dynamical behavior $(z''_4 - \zeta_4)$. We note that other relations such as $\mu = 2\zeta_2 - \zeta_4$ have been proposed in the literature.[14] L'vov and Procaccia[15] obtained the relation $\mu = 2-\zeta_6$ based on fusion rules for equal time multi-point correlation functions but they also pointed out another possible scenario which yields $\mu = 2\zeta_2 - \zeta_4$, the same exponent as in ref. 14.

The need for a hierarchy of dynamical exponents follows from the occurrence of temporal multiscaling in dynamical structure functions as has been emphasized earlier by L'vov *et al.*[10] in the quasi-Lagrangian formalism; thus different order temporal derivatives of $S_p(r, \tau)$ can lead to different dynamical exponents. The $\tau = 0$ limit of dynamical structure functions was also considered by us for the stochastic Burgers problem in ref. 11. In dealing with dynamical structure functions it is important to remember that we have used the Eulerian formalism of the Navier–Stokes equation; therefore, ordinary dynamic scaling and a fortiori, dynamic multifractality, are complicated by the presence of sweeping terms. For example, in $S_p(r = 0, \tau)$ (obtained from measurements of velocity differences at a given point at finite values of the time difference) the kinematic exponent $z = 1$ arising from sweeping occurs.[12] However, in the (Galilean invariant) convective derivative which occurs in Eq. (19) in the $\tau \to 0$ limit only the dynamical exponent occurs. Thus our equation for $\langle \epsilon\epsilon' \rangle$ with the right-hand side expressed in explicitly Galilean-invariant form neatly picks out the intrinsic dynamical exponent.

Our results on the two dominant intermittency exponents, one arising from the fourth order structure function, the second from a sixth-order structure function, only hold if the pressure terms do not give a more dominant contribution. We now discuss the pressure terms on the right-hand side of Eq. (19); in Appendix D we have displayed the equations satisfied by $\hat{D}^2\langle(\delta\vec{u}\cdot\delta\vec{u})^2\rangle/\hat{D}\tau^2$ and $\partial^2\langle\delta u_i\,\delta u_j(\delta\vec{u}\cdot\delta\vec{u})^2\rangle/\partial r_i\,\partial r_j$ which can be derived from the Navier–Stokes equation. The two pressure-dependent terms in the last line of Eq. (19) occur in these equations and hence, their behavior cannot be more dominant than the behavior we have deduced above. The remaining pressure terms involving a product of $\partial_i\tilde{p}$ and $\partial'_j\tilde{p}'$ in the second line of Eq. (19) are discussed in Appendix E. By inverting the equation satisfied by the pressure $\nabla^2\tilde{p} = -\partial_i\partial_j(u_iu_j)$ which arises from incompressibility we argue that these yield an exponent $\mu_1 = 2 - \zeta_6$ and nothing more dominant. The two pressure-dependent terms in the last line of Eq. (19) treated in Appendix D can also be analyzed using the methods of Appendix E and yield the same results. Thus, we have provided, we believe, persuasive arguments in Appendices D and E that the pressure terms do not yield more dominant behavior than that implied by the exponents μ_1 and μ_2 and hence, do not alter our result. The advantage of having the full equation for the dissipation-rate correlation which we have derived is that the structure of the additional terms involving pressure is explicit which should encourage further rigorous theoretical and detailed experimental investigations of their behavior in the inertial range.

In the absence of an explicit calculation of the exponents which characterize the spatial and temporal behavior of structure functions, one cannot decide which of the two exponents $\mu_1 = 2-\zeta_6$ and $\mu_2 = z''_4 - \zeta_4$ dominates the behavior of the dissipation rate correlations. We make the following additional comments motivated by a similar strategy in the theory of phase transitions, where as pointed out in ref. 13, an assertion of the equality of the (singular part) free-energy density and the inverse correlation volume yields the hyperscaling relations in critical phenomena. Thus we make an *ansatz* that the first two terms on the right-hand side of Eq. (19) are equally dominant. This leads to the identification

$$2-\zeta_6 = z''_4 - \zeta_4. \tag{20}$$

Thus by invoking this hypothesis of hyperscaling in the behavior of the equal-time dissipation rate correlations relation we obtain an interesting connection between multifractality in spatial correlations and multifractality in temporal correlations; in addition, we obtain the standard result $\mu = 2-\zeta_6$ which follows form the Refined Similarity Hypothesis.[1]

Finally, we comment on the conflicting experimental results, as discussed in ref. 5, obtained by measuring different subtracted, $\langle \epsilon\epsilon' \rangle - \langle \epsilon \rangle^2$, and unsubtracted, $\langle \epsilon\epsilon' \rangle$, dissipation rate correlations. In the inertial range we expect the term $(L/r)^{\mu}$ to dominate over the constant $\langle \epsilon \rangle^2$ term. However, the experimental determination of μ is rendered difficult by the smallness of μ. Only when $r \ll L$ does it not matter whether one uses the subtracted or unsubtracted dissipation rate correlation functions, since the term $\langle \epsilon\epsilon' \rangle$ dominates. This is clear from the experimental discussion in ref. 5. For r not much smaller than L the subtracted correlation function should be used; this is the form which arises naturally in our theoretical approach on the left hand side of the dissipation rate correlation equation, Eq. (19).

APPENDIX A

We outline details of the computation of the term which occurs on the left-hand side of Eq. (11), i.e., the term

$$\nu^2 \langle \delta u_i \; \delta u_j \; \partial_\alpha \partial_\alpha u_i \; \partial'_\beta \partial'_\beta u'_j \rangle$$

where we have used the notation $\partial_\alpha = \partial/\partial x_\alpha$, $\partial'_\beta = \partial/\partial x'_\beta$ and employed the summation convention. We will need the simple but key identity

$$\nu u_i \; \partial_\alpha \partial_\alpha u_i = \frac{\nu}{2} \partial_\alpha \partial_\alpha (\vec{u} \cdot \vec{u}) - \epsilon_{ii} \tag{A.1}$$

using the definition $\epsilon_{ij} = \nu\, \partial_\alpha u_i\, \partial_\alpha u_j$. We expand $\delta u_i\, \delta u_j$ to obtain four terms; we consider the case where we have $u_i u'_j$ and use the identity displayed above to obtain

$$\begin{aligned} &-\nu^2 \langle u_i\, \partial_\alpha \partial_\alpha u_i u'_j\, \partial'_\beta \partial'_\beta u'_j \rangle \\ &\quad = -\langle \epsilon_{ii} \epsilon'_{jj} \rangle + \nu\, \nabla_r^2 \left\langle \epsilon'_{jj}\, \frac{\vec{u}\cdot\vec{u}}{2} + \epsilon_{ii}\, \frac{\vec{u}'\cdot\vec{u}'}{2} \right\rangle - \frac{\nu^2}{4} (\nabla_r^2)^2 \langle \vec{u}\cdot\vec{u}\, \vec{u}'\cdot\vec{u}' \rangle. \end{aligned} \tag{A.2}$$

We observe that the terms in the second line consist of correlations which are finite in the inertial range multiplied by ν which tends to 0. We also obtain similarly for the term containing $u'_i u_j$,

$$\begin{aligned} &-\nu^2 \langle u_j\, \partial_\alpha \partial_\alpha u_i u'_i\, \partial'_\beta \partial'_\beta u'_j \rangle \\ &\quad = -\langle \epsilon_{ij} \epsilon'_{ij} \rangle + \frac{\nu}{2}\, \nabla_r^2 \langle \epsilon'_{ij} u_i u_j + \epsilon_{ij} u'_i u'_j \rangle + \nu^2\, \frac{\partial}{\partial r_\alpha} \frac{\partial}{\partial r_\beta} \langle u_j\, \partial_\alpha u_i u'_i\, \partial'_\beta u'_j \rangle. \end{aligned} \tag{A.3}$$

The other terms yield contributions which are negligible in the inertial range, as for example,

$$-\nu^2 \langle u_i u_j\, \partial_\alpha \partial_\alpha u_i\, \partial'_\beta \partial'_\beta u'_j \rangle = -\nu^2\, \nabla_r^2\, \frac{\partial}{\partial r_\alpha} \langle u_j u_i u'_j\, \partial_\alpha u_i \rangle + \nu \nabla_r^2 \langle u_j u'_j \epsilon_{ii} + u_i u'_j \epsilon_{ij} \rangle, \tag{A.4}$$

and a similar equation for the term with $u'_i u'_j$. These equations show that the only terms which survive in the inertial range are

$$-\langle \epsilon_{ii} \epsilon'_{jj} \rangle - \langle \epsilon_{ij} \epsilon'_{ij} \rangle. \tag{A.5}$$

We can handle the terms which arise in Eq. (12) in a similar fashion to obtain the results quoted in Eq. (13).

APPENDIX B

We provide a few of the details used in deriving Eq. (15) in this appendix. The derivation relies on some kinematic relations which can be obtained by using the independence of $u_i(\vec{x}, t)$ of $\vec{x}'$ and t', etc. and incompressibility. Note that since $\partial/\partial t = (1/2)\, \partial/\partial T + \partial/\partial \tau$ and the correlations are independent of T in the steady state we can replace $\partial/\partial \tau$ by $\partial/\partial t$ or by $-\partial/\partial t'$. Thus we find

$$\frac{\partial^2}{\partial \tau\, \partial r_i}\langle(\delta\vec{u}\cdot\delta\vec{u})^2\, u_i\rangle = -\left\langle \frac{\partial}{\partial t'}\frac{\partial}{\partial x_i}\left[(\delta\vec{u}\cdot\delta\vec{u})^2\, u_i\right]\right\rangle$$

$$= -\left\langle \frac{\partial}{\partial t'}\left[4\delta\vec{u}\cdot\delta\vec{u}\;\delta u_k\, \frac{\partial u_k}{\partial x_i}\, u_i\right]\right\rangle$$

$$= \left\langle 4\delta\vec{u}\cdot\delta\vec{u}\,\frac{\partial u'_k}{\partial t'}\, u_i\,\frac{\partial u_k}{\partial x_i} + 8\delta u_l\,\frac{\partial u'_l}{\partial t'}\,\delta u_k\, u_i\,\frac{\partial u_k}{\partial x_i}\right\rangle. \quad \text{(B.1)}$$

We perform a similar set of manipulations choosing to replace $\partial^2/\partial\tau\,\partial r_i$ by $\partial/\partial t\,\partial/\partial x'_i$ to obtain

$$\frac{\partial^2}{\partial \tau\, \partial r_i}\langle(\delta\vec{u}\cdot\delta\vec{u})^2\, u'_i\rangle = \left\langle 4\delta\vec{u}\cdot\delta\vec{u}\,\frac{\partial u_k}{\partial t}\, u'_i\,\frac{\partial u'_k}{\partial x'_i} + 8\delta u_l\,\frac{\partial u_l}{\partial t}\,\delta u_k\, u'_i\,\frac{\partial u'_k}{\partial x'_i}\right\rangle. \quad \text{(B.2)}$$

It is useful to note that the right-hand sides of the above two equations contain precisely parts of

$$C_1 = 2\left\langle \delta u_i\,\frac{Du_i}{Dt}\,\delta u_j\,\frac{Du'_j}{Dt'}\right\rangle + \left\langle \delta u_j\,\frac{Du_i}{Dt}\,\delta u_j\,\frac{Du'_i}{Dt'}\right\rangle$$

defined in Eq. (14) which we wish to evaluate. The other terms can be obtained by noting

$$\frac{\partial^2}{\partial \tau^2}\langle(\delta\vec{u}\cdot\delta\vec{u})^2\rangle = -\left\langle \frac{\partial}{\partial t'}\frac{\partial}{\partial t}(\delta\vec{u}\cdot\delta\vec{u})^2\right\rangle$$

$$= \left\langle 4\delta\vec{u}\cdot\delta\vec{u}\,\frac{\partial u'_i}{\partial t'}\frac{\partial u_i}{\partial t} + 8\delta u_i\,\frac{\partial u'_i}{\partial t'}\,\delta u_j\,\frac{\partial u_j}{\partial t}\right\rangle. \quad \text{(B.3)}$$

Along with a similar equation for

$$\frac{\partial^2}{\partial r_i \partial r_j}\langle(\delta\vec{u}\cdot\delta\vec{u})^2\,(u_i u'_j + u'_i u_j)\rangle$$

we obtain the result quoted in Eq. (15).

APPENDIX C

We discuss the evaluation of the noise terms and the terms involving $\langle\epsilon_{ij}\epsilon'_{ij}\rangle$ which occur in Eq. (17).

The noise terms can be evaluated using the Donsker–Novikov–Varadhan result for an arbitrary functional $\mathscr{F}$ of the velocity field, given the Gaussian noise characteristics in Eq. (5):

$$\langle \hat{f}_\alpha(\vec{k}, t)\, \mathscr{F}[\hat{u}(\vec{q}, t)] \rangle = \frac{1}{2}\, \hat{D}_{\alpha\beta}(\vec{k}) \left\langle \frac{\delta \mathscr{F}}{\delta \hat{u}_\beta(-\vec{k}, t)} \right\rangle. \tag{C.1}$$

We recall that the factor of $1/2$ occurs because we take the average of the limit in which the time variable in the noise approaches the time variable of the velocity field from above and below. In particular, with the notation employed in the paper we have

$$\langle f_i u'_j \rangle = 0 \qquad \text{and} \qquad \langle f'_i u_j \rangle = \tfrac{2}{3}\, \delta_{ij} \langle \epsilon \rangle$$

for $\tau \to 0^+$; the first equation above is evident since the noise acts at a later time. We will evaluate the noise-dependent terms which occur in Eqs. (11) and (12). First consider the term

$$\langle \delta u_i\, \delta u_i (\nu \nabla^2 u_j f'_j + \nu \nabla'^2 u'_j f_j) \rangle$$

which can be seen, using the Gaussian nature of the noise, to be equal to

$$2[\langle \delta u_i f'_j \rangle \langle \delta u_i\, \nu \nabla^2 u_j \rangle + \langle \delta u_i f_j \rangle \langle \delta u_i\, \nu \nabla'^2 u'_j \rangle].$$

Since $\langle \delta u_i\, \nu \nabla'^2 u'_j \rangle = \langle \epsilon \rangle = -\langle \delta u_i\, \nu \nabla^2 u_j \rangle$ and $\langle \delta u_i f'_j \rangle = \langle \delta u_i f_j \rangle$ this term vanishes. Similarly one can show that the other set of terms

$$2\langle \delta u_i\, \delta u_j (\nu \nabla^2 u_i f'_j + \nu \nabla'^2 u'_j f_i) \rangle$$

also vanishes. The non-zero contributions come from

$$\langle \delta u_i\, \delta u_i f_j f'_j \rangle = 2\langle \delta u_i f_j \rangle \langle \delta u_i f'_j \rangle = \tfrac{2}{3} \langle \epsilon \rangle^2 \tag{C.2}$$

and

$$\begin{aligned} 2\langle \delta u_i\, \delta u_j f_i f'_j \rangle &= 2\langle \delta u_i f_i \rangle \langle \delta u_j f'_j \rangle + 2\langle \delta u_i f'_j \rangle \langle \delta u_j f_i \rangle \\ &= 2\left[\langle \epsilon \rangle \langle \epsilon \rangle + \langle \epsilon \rangle \frac{\langle \epsilon \rangle}{3} \right] = \frac{8}{3} \langle \epsilon \rangle^2. \end{aligned} \tag{C.3}$$

Together these yield the term $10\langle \epsilon \rangle^2/3$ quoted in the text.

APPENDIX D

First we present the equation for the Galilean-invariant time derivative of the fourth-order structure function. The derivation is tedious and will not be presented here; the equation is given by

$$\begin{aligned}\frac{\hat{D}^2}{\hat{D}\tau^2}\langle(\delta\vec{u}\cdot\delta\vec{u})^2\rangle = &-2\,\frac{\hat{D}}{\hat{D}\tau}\langle\delta\epsilon_{ii}\,\delta\vec{u}\cdot\delta\vec{u}+2\delta\epsilon_{ij}\,\delta u_i\,\delta u_j\rangle+\text{noise terms}\\ &-2\,\frac{\partial}{\partial r_i}\langle(\epsilon_{ij}+\epsilon'_{ij})\,\delta u_j\,\delta\vec{u}\cdot\delta\vec{u}\rangle\\ &-\frac{1}{4}\frac{\partial}{\partial r_i}\langle(\partial_i p-\partial'_i p')(\delta\vec{u}\cdot\delta\vec{u})^2\rangle\\ &-2\,\frac{\hat{D}}{\hat{D}\tau}\langle(\partial_i p+\partial'_i p')\,\delta u_i\,\delta\vec{u}\cdot\delta\vec{u}\rangle.\end{aligned}\tag{D.1}$$

The crucial point is that the time derivative of the pressure terms which occur here are *precisely* those that occur in the final equation for the dissipation rate correlations in Eq. (19). Thus the behavior of this term is not more dominant than that implied by the second temporal derivative of the fourth-order structure function which we have discussed in the text.

Since the pressure terms which we need also involve other spatial derivatives, we need another equation to determine them. The equation for the first spatial derivative of the sixth-order structure function was given in our earlier paper[(7)] and we reproduce it here for completeness:

$$\begin{aligned}&\frac{\partial}{\partial r_j}\langle(\delta\vec{u}\cdot\delta\vec{u})^2\,\delta u_j\,\delta u_k\rangle\\ &\quad=\frac{\nu}{2}\nabla_r^2\langle[\delta\vec{u}\cdot\delta\vec{u}]^2\,\delta u_k\rangle-4\langle\delta\vec{u}\cdot\delta\vec{u}\,\delta u_k[\hat{\epsilon}+\hat{\epsilon}']\rangle-6\langle\delta\vec{u}\cdot\delta\vec{u}\,\delta u_i[\hat{\epsilon}_{ki}+\hat{\epsilon}'_{ki}]\rangle\\ &\quad\quad-2\langle\delta u_i\,\delta u_j\,\delta u_k[\hat{\epsilon}_{ij}+\hat{\epsilon}'_{ij}]\rangle\langle[\delta\vec{u}\cdot\delta\vec{u}]^2\,\delta\eta_k+4\delta u_k\,\delta\vec{u}\cdot\delta\vec{u}\,\delta\vec{u}\cdot\delta\vec{\eta}\rangle\\ &\quad\quad-2\left\langle[\delta\vec{u}\cdot\delta\vec{u}]^2\,\frac{\partial(p+p')}{\partial r_k}\right\rangle-4\left\langle\delta\vec{u}\cdot\delta\vec{u}\,\delta u_k\,\delta u_i\,\frac{\partial(p+p')}{\partial r_i}\right\rangle.\end{aligned}\tag{D.2}$$

Observing that $\partial(p+p')/\partial r_i=(1/2)(\partial_i p-\partial'_i p')$ and taking the derivative of the above equation with respect to r_k we find that the last term in the above equation is precisely the one that occurs in Eq. (19); the other term involving pressure in the above equation is of the same form as the term we need and using isotropy and homogeneity one can argue that they yield the same behavior. This provides the basis for our claim that the most dominant behavior arising from the two pressure terms considered in this appendix does not lead to behavior more dominant than those terms which yield the exponents μ_1 and μ_2 discussed in the main body of the paper.

APPENDIX E

In this appendix we discuss contributions in the basic equation for the dissipation-rate correlation, Eq. (17) which contain a product of pressure terms of the form $\partial_i \tilde{p}\, \partial'_j \tilde{p}'$. We recall the relation $\nabla^2 \tilde{p} = -\partial_i u_j\, \partial_j u_i$ obtained from the Navier–Stokes equation using incompressibility. Either from dimensional analysis of this equation or from the fact that for an incompressible flow the energy flux density is given by $\rho \vec{u}(u^2/2+p)$ we would expect terms such as $\langle \delta u_i\, \delta u_j\, \partial_i \tilde{p}\, \partial'_j \tilde{p}' \rangle$ to scale as the second derivative of a sixth-order structure function. However, writing the pressure in terms of the velocities involves long-ranged kernels and in this appendix we argue that nevertheless the terms do not contribute terms more dominant than the ones we have discussed. The relation for $\tilde{p}$ in terms of the velocities can be rewritten as $\nabla^2 \tilde{p} = -\partial_i \partial_j\, \delta u_i\, \delta u_j$, (where $\delta u_i = u_i - u'_i$), since $u'_i \equiv u_i(\vec{x}', t')$ is independent of $\vec{x}$. With a similar expression for the primed variables we can obtain derivatives of $\delta \tilde{p} = p - p'$. Since, $\vec{R} = (\vec{x}+\vec{x}')/2$ and $\vec{r} = \vec{x}-\vec{x}'$ we have $\partial/\partial R_i = \partial/\partial x_i + \partial/\partial x'_i$ and $\partial/\partial r_i = (1/2)(\partial/\partial x_i - \partial/\partial x'_i)$. Using these one can show that

$$\nabla_R^2 \delta \tilde{p} = \nabla^2 \tilde{p} - \nabla'^2 \tilde{p}' = -(\partial_i \partial_j - \partial'_i \partial'_j)\, \delta u_i\, \delta u_j = -2 \frac{\partial}{\partial R_i} \frac{\partial}{\partial r_j}\, \delta u_i\, \delta u_j. \tag{E.1}$$

This equation can be inverted formally using the Coulomb kernel to obtain

$$\delta \tilde{p} = \frac{1}{2\pi} \int d^3 R' \, \frac{1}{|\vec{R}-\vec{R}'|} \frac{\partial}{\partial R'_k} \frac{\partial}{\partial r_l}\, \delta u_k(\vec{R}', \vec{r})\, \delta u_l(\vec{R}', \vec{r}) \tag{E.2}$$

where we have $\delta u_k(\vec{R}', \vec{r}) \equiv u_k(\vec{R}'+\vec{r}/2) - u_k(\vec{R}'-\vec{r}/2)$ etc. From the above equation we obtain the useful result

$$\frac{\partial \delta \tilde{p}}{\partial R_i} = -\frac{1}{2\pi} \int d^3 R' \, \frac{\partial^2}{\partial R_i R'_k} \left[\frac{1}{|\vec{R}-\vec{R}'|} \right] \frac{\partial}{\partial r_l}\, \delta u_k(\vec{R}', \vec{r})\, \delta u_l(\vec{R}', \vec{r}). \tag{E.3}$$

The kernel defined by

$$K_{ik}(\vec{R},\vec{R}')=\frac{\partial^2}{\partial R_i R'_k}\left[\frac{1}{|\vec{R}-\vec{R}'|}\right] \tag{E.4}$$

is *quadrupolar* and decays more rapidly than a simple Coulombic form: for example, the integral over space of $K_{ij}(\vec{R})$ is conditionally convergent. It is straightforward to obtain, in a similar fashion, the following result for $\Delta\tilde{p}=\tilde{p}(\vec{R}+\vec{r}/2)+\tilde{p}(\vec{R}-\vec{r}/2)$:

$$\begin{aligned}\frac{\partial\Delta\tilde{p}}{\partial R_i}&=-\frac{1}{2\pi}\int d^3R'\,K_{ik}(\vec{R},\vec{R}')\,\frac{\partial}{\partial r_l}\\&\times[u_k(\vec{R}'+\vec{r}/2)\,u_l(\vec{R}'+\vec{r}/2)-u_k(\vec{R}'-\vec{r}/2)\,u_l(\vec{R}'-\vec{r}/2)]\end{aligned} \tag{E.5}$$

We use the expressions given above to investigate the pressure terms containing $\partial_i\tilde{p}\,\partial'_j\tilde{p}'$ in Eq. (19). Consider the term

$$\langle\delta u_i\,\delta u_j\,\partial_i\tilde{p}\,\partial'_j\tilde{p}'\rangle=\frac{1}{4}\left\langle\delta u_i\,\delta u_j\left(\frac{\partial\Delta\tilde{p}}{\partial R_i}\frac{\partial\Delta\tilde{p}}{\partial R_j}-\frac{\partial\delta\tilde{p}}{\partial R_i}\frac{\partial\delta\tilde{p}}{\partial R_j}\right)\right\rangle.$$

Using the above identities we have

$$\begin{aligned}&\left\langle\delta u_i\,\delta u_j\,\frac{\partial\delta\tilde{p}}{\partial R_i}\frac{\partial\delta\tilde{p}}{\partial R_j}\right\rangle\\&\quad\propto\int d^3R'\int d^3R''\,K_{ik}(\vec{R},\vec{R}')\,K_{jm}(\vec{R},\vec{R}'')\\&\quad\times\left\langle\delta u_i\,\delta u_j\,\frac{\partial}{\partial r_l}\,\delta u_k(\vec{R}',\vec{r})\,\delta u_l(\vec{R}',\vec{r})\,\frac{\partial}{\partial r_n}\,\delta u_m(\vec{R}'',\vec{r})\,\delta u_n(\vec{R}'',\vec{r})\right\rangle.\end{aligned} \tag{E.6}$$

The above expression involves integrals over multipoint velocity structure functions, a product of three different differences of velocities at points separated by the same $\vec{r}$, located, however, at $\vec{R}$, $\vec{R}'$, and $\vec{R}''$. We have to investigate the dominant contributions from the expectation values appropriately integrated over the quadrupolar kernels. First we note that

$$\frac{\partial^2}{\partial R_i R_j}\left[\frac{1}{|\vec{R}|}\right]=\frac{\delta_{ij}-3R_iR_j/R^2}{R^3}+\frac{4\pi}{3}\,\delta_{ij}\delta(\vec{R}).$$

The delta function terms yield in Eq. (E.6)

$$\left\langle\delta u_i\,\delta u_j\,\frac{\partial}{\partial r_l}\,\delta u_i\,\delta u_l\,\frac{\partial}{\partial r_n}\,\delta u_j\,\delta u_n\right\rangle$$

which is essentially of the form $\partial^2 S_6/\partial r^2$, a term obtained before. For large separations, i.e., $|\vec{R}'-\vec{R}|$ and $|\vec{R}''-\vec{R}| \gg r$ we expect the expectation value to factorize and these will either vanish (the angular quadrupolar integrals average out) or at most yield terms of lower order, $S_2\, \partial^2 S_4/\partial r^2$ when $\vec{R}' \approx \vec{R}''$. For $|\vec{R}'-\vec{R}|$ and $|\vec{R}''-\vec{R}| \approx r$, the quadrupolar kernels along with the volume element is dimensionless and the angular part causes the integrand to fluctuate yielding a contribution that will not dominate the delta function terms which unambiguously yield $\mu_1 = 2-\zeta_6$. The other integrals can be analyzed similarly. Thus we expect the pressure terms in Eq. (19) to yield terms no more dominant than the ones obtained by us from the pressure-independent contributions.

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REFERENCES

1. U. Frisch, *Turbulence* (Cambridge, New York, 1995).
2. A. S. Monin and A. M. Yaglom, *Statistical Fluid Mechanics: Mechanics of Turbulence* (MIT Press, Cambridge, 1975).
3. M. Nelkin, *Adv. Phys.* **43**:143–181 (1994).
4. G. Stolovitzky and K. R. Sreenivasan, *Rev. Mod. Phys.* **66**:229–240 (1994).
5. K. R. Sreenivasan and P. Kailasanath, *Phys. Fluids A* **5**:512–513 (1993).
6. The spatio-temporal behavior of $S_2(\vec{r}, \tau)$ was considered through an exact equation in S. Chandrasekhar, *Proc. Roy. Soc. (London)* **229**:1 (1955).
7. F. Hayot and C. Jayaprakash, *Phys. of Fluids* **12**:327–335 (2000).
8. L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Pergamon, New York, 1986); note the neglect of the term $\nu\, d^2S_2/dr^2$ in the inertial range on p. 140.
9. See p. 43 of Frisch in ref. 1.
10. V. S. L'vov, E. Podivilov, and I. Procaccia, *Phys. Rev. E* **55**:7030–7035 (1997).
11. F. Hayot and C. Jayaprakash, *Phys. Rev. E* **56**:227 (1997) where the $\tau = 0$ limit of dynamical structure functions was considered.
12. H. Tennekes, *J. Fluid Mech.* **67**:561–567 (1975).
13. See, M. E. Fisher, in *Collective Properties of Physical Systems*, B. Lundqvist and S. Lundqvist, eds. (Academic, N.Y., 1974), p. 23.
14. M. Nelkin and T. L. Bell, *Phys. Rev. A* **17**:363–369 (1978). See also, U. Frisch, P.-L. Sulem, and M. Nelkin, *J. Fluid Mech.* **87**:719–736 (1978).
15. V. L'vov and I. Procaccia, *Phys. Rev. Lett.* **76**:2898–2901 (1996); *Phys. Rev. E* **53**: 3468–3490 (1995); *Phys. Rev. Lett.* **77**:3541–44 (1996).